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Studying the Bell–Steinberger relation

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Abstract. The Bell–Steinberger relation is analyzed. The questionable points of the standard derivation of this relation are discussed. It is shown that the use of a more accurate approximation than the one usually used in the derivation of this relation can lead to corrections to the right hand side of the standard Bell–Steinberger relation.

1 Introduction

The Bell–Steinberger (BS) unitary relation [1,2] is considered as a very useful and effective tool in searching for properties of the $K_0, \overline{K}_0$ subsystem [3–15]. Some tests of the fundamental CPT-and T-invariance [16] are based on the BS relation [3,5,6,10–15]. The BS relation holds in the approximate Lee–Oehme–Yang (LOY) theory of time evolution in the neutral kaon subsystem [17–21], which follows from the Weisskopf–Wigner (WW) approximation [22]. Khalfin [8,9] has shown that the BS relation in its original form is not true in the exact theory. A similar conclusion can be drawn from the result contained in [23]. This means that the interpretation of the results of all the tests in the neutral kaon subsystem, which are based on the BS relation, cannot be considered as ultimate. The proper interpretation of such tests is impossible without a detailed investigation of the weak points of this relation.

The original (standard) form of the BS relation is the following:

$$\left[\frac{\gamma_s + \gamma_l}{2} - \mathrm{i}(m_s - m_l)\right] \langle s|l\rangle = \sum_F \langle F|T|s\rangle^* \langle F|T|l\rangle. \quad (1)$$

The derivation of this relation in such form is possible if the transition operator T exists [1,2]. It is assumed there that the operator T describes transitions from states belonging to the subspace, say $\mathcal{H}_{||}$, of states of neutral kaons into the subspace of their decay products, $\mathcal{H}_\perp$. Here $|l\rangle$, $|s\rangle \in \mathcal{H}_{||}$, $|F\rangle \in \mathcal{H}_\perp$ and $\{|F\rangle\}$ forms a complete orthonormal set in $\mathcal{H}_\perp$. The Hilbert space $\mathcal{H} = \mathcal{H}_{||} \bigoplus \mathcal{H}_\perp$ is the state space of the total system under consideration. The vectors $|l\rangle, |s\rangle$ are the normalized eigenvectors of the effective Hamiltonian, $H_{||}$, for the neutral K mesons complex, for the eigenvalues $\mu_{l(s)} = m_{l(s)} - \frac{\mathrm{i}}{2}\gamma_{l(s)}$ respectively,

$$H_{||}|l(s)\rangle = \mu_{l(s)}|l(s)\rangle. \quad (2)$$

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We have

$$|l\rangle = N_l(p_l|\mathbf{1}\rangle - q_l|\mathbf{2}\rangle), \;\; |s\rangle = N_s(p_s|\mathbf{1}\rangle + q_s|\mathbf{2}\rangle). \quad (3)$$

Here $|\mathbf{1}\rangle$ stands for vectors of the $|K_0\rangle$, $|B_0\rangle$, $|n\rangle$ (for neutrons) type et cetera, and $|\mathbf{2}\rangle$ denotes antiparticles of the particle "1": $|\overline{K}_0\rangle$, $|\overline{B}_0\rangle$, $|\overline{n}\rangle$, and so on, $\langle\mathbf{j}|\mathbf{k}\rangle = \delta_{jk}$, $j,k = 1,2$.

The linear operator $H_{||}$ is the (2×2) non-hermitian matrix,

$$H_{||} \equiv \begin{pmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{pmatrix} = M - \frac{\mathrm{i}}{2}\,\Gamma, \quad (4)$$

(where $M = M^+$ is the mass matrix and

$$\Gamma = \Gamma^+ \equiv \mathrm{i}(H_{||} - H_{||}^+) \quad (5)$$

denotes the decay matrix), acting in $\mathcal{H}_{||}$. The operators M and Γ are linear. From (5) and (4) one finds that

$$\Gamma_{jk} = \mathrm{i}(h_{jk} - h_{kj}^*) \quad (j,k = 1,2). \quad (6)$$

2 Derivation of the standard Bell–Steinberger relation

In deriving the relation (1) one usually invokes the probability conservation [1,2]. Probability conservation means that for every vector $|\psi;t\rangle \in \mathcal{H}$ solving the Schrödinger equation

$$\mathrm{i}\frac{\partial}{\partial t}|\psi;t\rangle = H|\psi;t\rangle, \quad (7)$$

one has

$$||\,|\psi;t\rangle\,||^2 = 1, \quad (8)$$

for every t. In the case considered the condition (8) can be rewritten as follows:

$$||\,|\psi;t\rangle_{||}\,||^2 \;+\; ||\,|\psi;t\rangle_\perp\,||^2 \;=\; 1, \quad (9)$$

where

$$|\psi; t\rangle_{||} \stackrel{\text{def}}{=} P|\psi; t\rangle \in \mathcal{H}_{||}, \quad |\psi; t\rangle_{\perp} \stackrel{\text{def}}{=} Q|\psi; t\rangle \in \mathcal{H}_{\perp}, \quad (10)$$

and P is the projection operator onto the subspace $\mathcal{H}_{||}$:

$$P \equiv |\mathbf{1}\rangle\langle\mathbf{1}| + |\mathbf{2}\rangle\langle\mathbf{2}|, \quad (11)$$

Q is the projection operator onto the subspace of decay products $\mathcal{H}_{\perp}$, $Q \equiv I - P$.

The initial condition for the Eq. (7) for the problem considered is

$$|\psi\rangle \stackrel{\text{def}}{=} |\psi; t = 0\rangle \equiv P|\psi\rangle \stackrel{\text{def}}{=} |\psi\rangle_{||} \in \mathcal{H}_{||}. \quad (12)$$

From (9) it follows that

$$-\frac{\partial}{\partial t} || \, |\psi; t\rangle_{||} \, ||^2 \; = \; \frac{\partial}{\partial t} || \, |\psi; t\rangle_{\perp} \, ||^2, \quad (13)$$

Using this relation one usually assumes that its right hand side can be identified with the following expression:

$$\frac{\partial}{\partial t} || \, |\psi; t\rangle_{\perp} \, ||^2 \stackrel{(?)}{=} \sum_F (\langle F|T|\psi; t\rangle_{||})^* \, \langle F|T|\psi; t\rangle_{||}, \quad (14)$$

and thus, without proving if (or when) the relation (14) is true, one finds that

$$-\frac{\partial}{\partial t} || \, |\psi; t\rangle_{||} \, ||^2 \; \equiv \; \sum_F (\langle F|T|\psi; t\rangle_{||})^* \, \langle F|T|\psi; t\rangle_{||}, \quad (15)$$

which leads to original BS formula (1). Indeed, following [1] and inserting

$$|\psi; t\rangle_{||} \; = \; x \, \mathrm{e}^{-\mathrm{i}t\mu_l} |l\rangle \; + \; y \, \mathrm{e}^{-\mathrm{i}t\mu_s} |s\rangle, \quad (16)$$

into (15) (where x, y are arbitrary time-independent number coefficients), and differentiating with respect to t the left hand side of the relation (15) and then putting $t = 0$, one obtains all the relations derived in [1]. Of course such a derivation is possible if μ_l, μ_s do not depend on time t.

Note that such a method of derivation of the BS relation (1) cannot be considered as rigorous and correct. Namely, it requires the existence of the transition operator T for the particles under study. Next, the decay of the states under investigation must be described by the exponential function of time t for all times t. Such an assumption is not consistent with the fundamental properties of quantum evolution. From the properties of solutions of the Schrödinger equation and from the basic principles of quantum theory it follows that the decay process cannot be exponential for times $t \to 0$ and for times $t \to \infty$ [24]. What is more, in [25] it was shown that the CPT theorem of axiomatic quantum field theory is not valid in a system containing exponentially decaying particles. This means simply that the BS relation (1) derived in such a way cannot be used for designing CPT-violation tests.

The last weak point of this derivation of the relation (1) is the following inconsistency. Using the Schrödinger equation it is not difficult to verify that

$$\frac{\partial}{\partial t} || \, |\psi; t\rangle_{||} \, ||^2 \Big|_{t=0} \; \equiv \; 0, \quad (17)$$

for the arbitrary initial condition $|\psi\rangle \in \mathcal{H}$. Thus the relation (15) takes the following form at $t = 0$:

$$\begin{aligned} -\frac{\partial}{\partial t} || \, |\psi; t\rangle_{||} \, ||^2 \Big|_{t=0} \; &\equiv \; 0 \\ &= \sum_F (\langle F|T|\psi\rangle_{||})^* \, \langle F|T|\psi\rangle_{||} \; \neq 0. \end{aligned} \quad (18)$$

So the left hand side of the relation (1) equals zero whereas the right hand side of this relation is non-zero at $t = 0$. The conclusion is that one should be very careful using the original BS relation (1) as a tool for searching for properties of the neutral kaon and similar complexes.

In the original form of the BS relation, [1–15], the vectors $|l\rangle$ and $|s\rangle$ are understood as the eigenvectors for the LOY effective Hamiltonian, H_{LOY}. That is, $H_{||} \equiv H_{\mathrm{LOY}}$ in such a case.

3 Approximate effective Hamiltonians and the BS relation

In many papers the observation was made that in order to obtain the left hand side of the BS relation (1) one need not use the method based on the relations (14)–(16). It appears that the equivalent relation can be derived directly from the eigenvalue equation (2) (see, e.g., [12–14, 18]). Indeed, directly from (2) one finds

$$\left[\frac{\gamma_s + \gamma_l}{2} - \mathrm{i}(m_s - m_l)\right] \langle s|l\rangle = \langle s|\varGamma|l\rangle, \quad (19)$$

which within the LOY approximation is equivalent to (1).

This method of the derivation of the BS relation is free of the above mentioned inconsistencies. It has an advantage over the original one [1,2] because one does not make use of the transition matrix T. Simply, the assumption about the existence of the T operator is unnecessary in this case. It is a very important property of this method because, in fact, the correct definition of the scattering matrix, $S \equiv I + \mathrm{i}T$, and thus the T-matrix, makes use of asymptotic states. Such states do not exist for unstable particles and $K_0, \overline{K}_0$ mesons are unstable. What is more: within this method one need not assume that the decay is exponential.

The accuracy of the relation (19) is determined by the accuracy of the approximation leading to the $H_{||}$ used there. If one inserts into the eigenvalue equation (2) the effective Hamiltonian $H_{||} \equiv H_{\mathrm{LOY}}$, then one comes to the picture equivalent to the original BS treatment of this problem. On the other hand, if one uses the exact effective Hamiltonian $H_{||}$ then the relation (19) will not describe the approximate one but it will describe the real properties of the system under consideration. The use of a more accurate approximation for $H_{||}$ than the LOY approximation in (2), and thus in (19), will lead to a description of the system considered, which can be sensitive to possible effects unreachable by means of the LOY method.

The LOY effective Hamiltonian, H_{LOY}, can be expressed in a compact form as follows [27]:

$$H_{\mathrm{LOY}} = m_0 P - \Sigma(m_0) = M^{\mathrm{LOY}} - \frac{\mathrm{i}}{2}\Gamma^{\mathrm{LOY}}, \tag{20}$$

where

$$\Sigma(\epsilon) = PHQ\frac{1}{QHQ - \epsilon - \mathrm{i}0}QHP = \Sigma^R(\epsilon) + \mathrm{i}\Sigma^{\mathrm{I}}(\epsilon), \tag{21}$$

and $\Sigma^R(\epsilon = \epsilon^*) = \Sigma^R(\epsilon = \epsilon^*)^+$, $\Sigma^{\mathrm{I}}(\epsilon = \epsilon^*) = \Sigma^{\mathrm{I}}(\epsilon = \epsilon^*)^+$. The operator $\Sigma^{\mathrm{I}}(\epsilon)$ we are especially interested in has the following form:

$$\Sigma^{\mathrm{I}}(\epsilon) \equiv \pi\, PHQ\,\delta(QHQ - \epsilon)\,QHP. \tag{22}$$

Within the LOY approach the vectors $|\mathbf{1}\rangle$, $|\mathbf{2}\rangle$ are normalized eigenstates of the free Hamiltonian, $H^{(0)}$ (here $H \equiv H^{(0)} + H_{\mathrm{I}}$ is the total Hamiltonian of the system considered), for a 2-fold degenerate eigenvalue m_0:

$$H^{(0)}|\mathbf{j}\rangle = m_0|\mathbf{j}\rangle \quad (j = 1, 2). \tag{23}$$

H_{I} denotes the interaction which is responsible for the decay process.

From (20) one finds that

$$\Gamma^{\mathrm{LOY}} = 2\Sigma^{\mathrm{I}}(m_0), \tag{24}$$

which means that

$$\Gamma^{\mathrm{LOY}}_{jk} = 2\pi\langle\mathbf{j}|HQ\delta(QHQ - m_0)QH|\mathbf{k}\rangle, \tag{25}$$

$$\equiv \pi\sum_F \delta(E_F - m_0)\langle\mathbf{j}|PH|F\rangle\langle F|HP|\mathbf{k}\rangle, \tag{26}$$

where $j, k = 1, 2$, $H|F\rangle = E_F|F\rangle$ and $\sum_F |F\rangle\langle F| \equiv Q$. So within the LOY approximation using (26) one finds that in the CPT-invariant system the right hand side of the relation (19) takes the following form:

$$\langle s|\Gamma^{\mathrm{LOY}}|l\rangle = 2\langle s|\Sigma^{\mathrm{I}}(m_0)|l\rangle \tag{27}$$

$$\equiv 2\pi\sum_F \delta(E_F - m_0)\langle F|HP|s\rangle^*\,\langle F|HP|l\rangle. \tag{28}$$

This is the standard picture which one meets in the literature. Note that this expression coincides with the right hand side of (1).

Now, if one uses the exact $H_{||}$ instead of the approximate H_{LOY} in the relation (19) then one can expect that the right hand side of the BS relation (19) will differ from (28). The exact effective Hamiltonian $H_{||}$ is time dependent [2,23,25–34], $H_{||} = H_{||}(t)$, in the non-trivial case, and can be expressed as follows [23,25–34]:

$$H_{||} = PHP + V_{||}(t), \tag{29}$$

where the non-hermitian operator $V_{||}(t)$ has the following property:

$$V_{||}(t = 0) \equiv 0. \tag{30}$$

(The "non-trivial case" is understood here as the $[P, H] \neq 0$ case.) In the non-trivial case the property that the effective Hamiltonian depends on time, $H_{||} = H_{||}(t)$, has the following consequence: In a CPT-invariant but CP-non-invariant system the diagonal matrix elements h_{jj} ($j = 1, 2$), cannot be equal for $t > 0$ [23] and all coefficients $p_s, p_l, q_s, \ldots$ appearing in the formula (3) are different and time dependent [8,9,26]. The same is true for the eigenvalues μ_l, μ_s.

From (29) it follows that the matrix Γ can be expressed as follows:

$$\Gamma \equiv \Gamma(t) = \mathrm{i}\left(V_{||}(t) - (V_{||}(t))^+\right). \tag{31}$$

So, the relation (30) means that in the exact case $\Gamma(t = 0) = 0$. From this property and from the properties of the eigenvectors $|l\rangle = |l^t\rangle$ and $|s\rangle = |s^t\rangle$ [26,34] one concludes that at the initial instant $t = 0$ the BS relation (19) becomes trivial: $0 = 0$. It contrasts with the relation (18) and it is consistent with the basic assumptions of quantum theory. This is the simplest general conclusion which can be obtained for the exact case. For the considered models of interactions leading to the decay of $K_0, \overline{K}_0$ mesons it is practically impossible to calculate the exact effective Hamiltonian $H_{||}$. Nevertheless, one can study the BS relation using the more accurate approximate effective Hamiltonians $H_{||}$ than H_{LOY} and thus one can look for possible deviations from the standard (i.e., LOY) picture. An example of the $H_{||}$, more accurate than H_{LOY}, is given in [25–29].

The approximate formulae for $H_{\|}(t)$ have been derived there using the Krolikowski–Rzewuski (KR) equation for the projection of a state vector [35], which results from the Schrödinger equation (7) for the total system under consideration, and, in the case of initial conditions of the type (12), takes the following form:

$$\left(\mathrm{i}\frac{\partial}{\partial t} - PHP\right)U_{\|}(t)|\psi\rangle_{||} = -\mathrm{i}\int_0^\infty K(t-\tau)U_{\|}(\tau)|\psi\rangle_{||}\mathrm{d}\tau, \tag{32}$$

where $U_{\|}(0) = P$, and $U_{||}(t)$ is the evolution operator for the subspace $\mathcal{H}_{||}$,

$$K(t) = \Theta(t)PHQ\exp(-\mathrm{i}tQHQ)QHP, \tag{33}$$

and $\Theta(t) = \{1 \text{ for } t \geq 0,\ \ 0 \text{ for } t < 0\}$.

The integro-differential equation (32) is equivalent to the following differential one [30–35]:

$$\left(\mathrm{i}\frac{\partial}{\partial t} - H_{||}(t)\right)U_{\|}(t)|\psi\rangle_{||} = 0, \tag{34}$$

where the effective Hamiltonian $H_{||}(t)$ has the form (29). Taking into account (32), (34) and (29) one finds from (32)

$$V_{\|}(t)U_{\|}(t) = -\mathrm{i}\int_0^\infty K(t-\tau)U_{\|}(\tau)\mathrm{d}\tau. \tag{35}$$

This relation can be used to obtain the approximate formula for $V_{||}(t)$. From (35) one finds to the lowest non-trivial order [26,33]

$$V_{\|}(t) \cong V_{\|}^{(1)}(t) \stackrel{\text{def}}{=} -\mathrm{i}\int_0^{\infty} K(t-\tau)\exp\left[\mathrm{i}(t-\tau)PHP\right]\mathrm{d}\tau. \quad (36)$$

The use of P defined by the relation (11) leads to [27,28]

$$V_{||}(t) = -\frac{1}{2}\Xi(H_0+\kappa;t)\left[\left(1-\frac{H_0}{\kappa}\right)P + \frac{1}{\kappa}PHP\right] -\frac{1}{2}\Xi(H_0-\kappa;t)\left[\left(1+\frac{H_0}{\kappa}\right)P - \frac{1}{\kappa}PHP\right], \quad (37)$$

where

$$H_0 = \frac{1}{2}(H_{11}+H_{22}),\quad \kappa = \sqrt{|H_{12}|^2 + \frac{1}{4}(H_{11}-H_{22})^2}, \quad (38)$$

and

$$H_{jk} = \langle \mathbf{j}|H|\mathbf{k}\rangle \quad (j,k=1,2), \quad (39)$$

$$\Xi(x;t) \stackrel{\text{def}}{=} PHQ\frac{\mathrm{e}^{-\mathrm{i}t(QHQ-x)}-1}{QHQ-x}QHP. \quad (40)$$

Note that $V_{\|}(t) \cong V_{\|}^{(1)}(t) = 0$ at $t=0$, which agrees with the general property of the exact $H_{||}(t)$ and $V_{||}(t)$ (see (30)). The expression (37) leads by (31) to a very complicated form of $\Gamma(t)$. Such a form of Γ is very hard to compare with Γ^{LOY} and thus to relate it to the right hand side of the original BS relation (1). The form (37) of $V_{||}(t)$ becomes simpler when

$$\kappa \ll H_0, \quad (41)$$

because then

$$\Xi(H_0\pm\kappa;t) \simeq \Xi(H_0;t) \pm \kappa\left.\frac{\partial\Xi(x;t)}{\partial x}\right|_{x=H_0} + \dots \quad (42)$$

So, if the condition (41) holds, then

$$V_{||}(t) \simeq -\Xi(H_0;t) - \left.\frac{\partial\Xi(x;t)}{\partial x}\right|_{x=H_0}(PHP - H_0P) + \dots \quad (43)$$

(Note that due to the presence of resonance terms the second term on the right hand side of the above expression, that is $\frac{\partial\Xi(x;t)}{\partial x}$, need not be small). This last expression is simpler than (37) but it also leads to a time dependent Γ and thus $\langle s|\Gamma(t)|l\rangle$. Such a $\langle s|\Gamma(t)|l\rangle$ cannot be compared with the BS relation (1), which is applied for asymptotic times $t\to\infty$. One needs $V_{||}(t)$ for times t which are at least of the order of the lifetimes, τ_l, τ_s for states $|l\rangle, |s\rangle$, that is for $t\sim\tau_l$. It can be achieved using $V_{||} \stackrel{\text{def}}{=} \lim_{t\to\infty} V_{||}^{(1)}(t)$ instead of (37). We have

$$\lim_{t\to\infty}\Xi(x;t) = \Sigma(x). \quad (44)$$

Thus

$$V_{||} \stackrel{\text{def}}{=} \lim_{t\to\infty} V_{||}^{(1)}(t) = -\frac{1}{2}\Sigma(H_0+\kappa)\left[\left(1-\frac{H_0}{\kappa}\right)P + \frac{1}{\kappa}PHP\right] -\frac{1}{2}\Sigma(H_0-\kappa)\left[\left(1+\frac{H_0}{\kappa}\right)P - \frac{1}{\kappa}PHP\right]. \quad (45)$$

To realize the purpose of this paper it is sufficient to consider the case (41). So, if condition (41) holds then

$$\Sigma(H_0\pm\kappa) \simeq \Sigma(H_0) \pm \kappa\left.\frac{\partial\Sigma(x)}{\partial x}\right|_{x=H_0} + \dots \quad (46)$$

which, by (45), yields

$$V_{||} \simeq -\Sigma(H_0) - \left.\frac{\partial\Sigma(x)}{\partial x}\right|_{x=H_0}(PHP - H_0P) + \dots . \quad (47)$$

Thus, taking into account (24), (29) and (31), one finds that if the condition (41) holds, then

$$\Gamma \equiv \Gamma^{(0)} + \Delta\Gamma, \quad (48)$$

where

$$\Gamma^{(0)} = 2\,\Sigma^{\mathrm{I}}(H_0), \quad (49)$$

and

$$\Delta\Gamma = -\mathrm{i}\left[\left.\frac{\partial\Sigma(x)}{\partial x}\right|_{x=H_0}(PHP - H_0P) - (PHP-H_0P)\left.\left(\frac{\partial\Sigma(x)}{\partial x}\right)^{+}\right|_{x=H_0}\right]. \quad (50)$$

Thus in this case

$$\langle s|\Gamma|l\rangle \simeq \langle s|\Gamma^{(0)}|l\rangle + \langle s|\Delta\Gamma|l\rangle, \quad (51)$$

which evidently differs from (27).

Note that all the above discussed expressions (37)–(50) have been derived without assuming any symmetries of the type CP, T, or CPT for the total Hamiltonian H of the system considered. Now let us assume that the CPT-symmetry is conserved, that is

$$[\Theta, H] = 0, \quad (52)$$

where Θ is the antiunitary operator: $\Theta \stackrel{\text{def}}{=} \mathcal{CPT}$ and $\mathcal{C}$ denotes the charge conjugation, $\mathcal{P}$ is the space inversion (parity) and $\mathcal{T}$ the time reversal transformation. Let us assume also that the subspace of neutral kaons $\mathcal{H}_{\|}$ is invariant under Θ:

$$[\Theta, P] = 0. \quad (53)$$

When these two last assumptions hold then $H_{11} = H_{22}$, $\kappa \equiv |H_{12}|$ and $H_0 \equiv H_{11} \equiv H_{22}$ and also $\Sigma_{11}(\varepsilon=\varepsilon^*) \equiv \Sigma_{22}(\varepsilon=\varepsilon^*) \stackrel{\text{def}}{=} \Sigma_0(\varepsilon=\varepsilon^*)$. So, when the total system is CPT-invariant all the expressions for the approximate $V_{||}(t)$, (37) and (43), $V_{||}$, (45) and (47), and Γ, $\Gamma^{(0)}$, (49) and (50), become simpler. It is very important that this approximate $V_{||}$ leads to an effective Hamiltonian $H_{||}$ possessing properties consistent with the properties of the exact effective Hamiltonian: Analogously to the properties of the exact effective Hamiltonian [23] its diagonal matrix elements are not equal if the total system under consideration is CPT-invariant but CP-non-invariant [25,26,34]. This property is absent in the LOY approximation and therefore the approach based on the LOY effective Hamiltonian is unable to reflect correctly all the properties of the real system. So the description of the properties of the $K_0, \overline{K}_0$ complex within the use of the above described approximation based on the KR equation should lead to a more realistic picture of the behavior of this complex than that given by the LOY and related approaches.

In the case of preserved CPT-symmetry, i.e., when the conditions (52) and (53) hold, one can identify H_0 appearing in (49) with m_0 from the formula (24): $H_0 \equiv m_0$. This means that when the total system preserves CPT symmetry and the condition (41) holds, then

$$\varGamma^{(0)} \equiv \varGamma^{\rm LOY}. \tag{54}$$

Note that in this case still $\Delta\varGamma \neq 0$ (see (50)). Therefore one again finds that

$$\begin{aligned}\langle s|\varGamma|l\rangle &\simeq \langle s|\varGamma^{(0)}|l\rangle + \langle s|\Delta\varGamma|l\rangle \\ &\equiv \langle s|\varGamma^{\rm LOY}|l\rangle + \langle s|\Delta\varGamma|l\rangle \\ &\neq \langle s|\varGamma^{\rm LOY}|l\rangle. \end{aligned}\tag{55}$$

One observes from (43) and (47) (or (50)) that if the total Hamiltonian H has the following property:

$$PHP \equiv H_0\, P, \tag{56}$$

then simply

$$V_{||} = -\Sigma(H_0), \tag{57}$$

and $\Delta\varGamma \equiv 0$. So, in such a case

$$\varGamma \equiv \varGamma^{\rm LOY}, \tag{58}$$

and $H_{||} \equiv H_{\rm LOY}$.

The solution of the condition (56) is simple:

$$H_{12} = H_{21} = 0. \tag{59}$$

This means (by (39)) that if the first order $|\Delta S| = 2$ transitions do not occur in the $K_0, \overline{K}_0$ complex then the BS relation (19) with $\varGamma \equiv \varGamma^{\rm LOY}$ and with $\varGamma$ given by (31) and (48) coincide. This also means that the original BS relation should not be used for designing, e.g., tests verifying the existence of such interactions.

4 Final remarks

Note that from the relation (28), using the Schwartz inequality, the conclusion of the following type is drawn in the literature (see [1,18] and also see (6.85) in [7], et cetera):

$$\begin{aligned}|\langle s|\varGamma|l\rangle|^2 &\equiv |\pi \sum_F \delta(E_F - m_0)\langle F|HP|s\rangle^*\, \langle F|HP|l\rangle|^2 \\ &\leq \pi \sum_F \delta(E_F - m_0)\langle F|HP|s\rangle^*\, \langle F|HP|s\rangle \\ &\quad \times \pi \sum_F \delta(E_F - m_0)\langle F|HP|l\rangle^*\, \langle F|HP|l\rangle. \end{aligned}\tag{60}$$

This inequality is interpreted as follows:

$$|\langle s|\varGamma|l\rangle|^2 \leq \varGamma_s\, \varGamma_l, \tag{61}$$

where the decay widths $\varGamma_s, \varGamma_l$ are identified with

$$\varGamma_{l(s)} = \pi \sum_F \delta(E_F - m_0)\langle F|HP|l(s)\rangle^*\, \langle F|HP|l(s)\rangle. \tag{62}$$

The inequality (61) together with the BS relation (19) is used, e.g., to estimate the product $|\langle s|l\rangle|$ (see e.g., [1, 7,18]). In the light of the relations (48) and (55) such estimations and similar conclusions can be considered as consistent with the real properties of the system under investigation only if we would have $\varGamma \equiv \varGamma^{\rm LOY}$.

Note that the inequality (60) need not be true in the case when the relation (48) (or when the earlier expressions for $V_{||}$) holds. Then simply $\varGamma \neq \varGamma^{\rm LOY}$. This means that the estimations of type (61) need not be true in such a case. So, keeping in mind the relations (48) and (55) one should be very careful while considering the tests performed in the $K_0, \overline{K}_0$ complex within the use of the original BS relations (1) (or (19), where $\varGamma = \varGamma^{\rm LOY}$) as the crucial one.

On the other hand, due to the properties of the matrix $\varGamma$ ($\varGamma$ is a linear and hermitian matrix) the expression $\langle s|\varGamma|l\rangle$ defines the hermitian form in the subspace $\mathcal{H}_{||}$. Indeed, for every $|\phi\rangle, |\psi\rangle \in \mathcal{H}_{||}$ one can define

$$(\phi, \psi) \stackrel{\rm def}{=} \langle \psi|\varGamma|\phi\rangle. \tag{63}$$

Now, if the matrix $\varGamma$ is positive definite then the form (63) is a positive definite hermitian form. It is not difficult to verify that the form (ϕ, ψ) must then fulfill all the requirements of the scalar product. Therefore in this case for the product (ϕ, ψ) the Schwartz inequality holds, which reads

$$|(\phi, \psi)|^2 \leq (\phi, \phi)\, (\psi, \psi). \tag{64}$$

Within the use of the definition (63) this inequality can be rewritten as follows:

$$|\langle \psi|\varGamma|\phi\rangle|^2 \leq \langle \psi|\varGamma|\psi\rangle\, \langle \phi|\varGamma|\phi\rangle. \tag{65}$$

This inequality is true for every $|\psi\rangle, |\phi\rangle \in \mathcal{H}_{||}$ only if $\varGamma$ is positive definite.

Now if the eigenvectors $|s\rangle, |l\rangle$ of $H_{||}$ are inserted into (65) then one can find that

$$|\langle s|\varGamma|l\rangle|^2 \leq \langle s|\varGamma|s\rangle\, \langle l|\varGamma|l\rangle. \tag{66}$$

Thus, using the eigenvalue equation (2) for $H_{||}$ and the relation (5) one can conclude that

$$|\langle s|\varGamma|l\rangle|^2 \leq \gamma_s\, \gamma_l. \tag{67}$$

One should stress that γ_s, γ_l appearing in this inequality are determined by the solutions of the eigenvalue problem for $H_{||}$, but not by the relations (25), (26) or (62).

Unfortunately there has not been published any rigorous proof that the exact Γ should be positive definite. This is only a supposition following from the assumption that a decay process is considered and therefore suitable transition probabilities should be decreasing functions of time t and they should vanish for $t \to \infty$. What is more, Γ can be positive definite only if all such transition probabilities are monotically decreasing functions of t, which needs not be true in the general case. Of course Γ calculated within the LOY approximation is positive definite, but such a property of the LOY effective Hamiltonian cannot be considered as a general rigorous proof for the exact case. So the inequality (64) (and therefore the inequalities (65)–(67)) cannot be considered as definitely valid. What is more, there exist reasons leading to the conclusion that the matrix $\Delta\Gamma$ defined by the formula (50) is not positive and thus the matrix Γ connected with $\Delta\Gamma$ by the relation (48) need not be positive definite. Such a conclusion follows from the generalized Fridrichs–Lee model [36] and calculations performed in [26,25]. Results obtained there lead to the following form of $\Gamma = \Gamma^{\rm FL}$ in the CPT-invariant case:

$$\Gamma^{\rm FL} \equiv \begin{pmatrix} \gamma_{11} & \gamma_{12} \\ \gamma_{21} & \gamma_{22} \end{pmatrix} = \Gamma^{(0)} + \Delta\gamma, \qquad (68)$$

where $\Gamma^{(0)} = \Gamma^{\rm LOY}$, $\gamma_{jk} = \Gamma^{\rm LOY}_{jk} + \Delta\gamma_{jk}$ $(j,k = 1,2)$ and

$$\Delta\gamma_{11} = \Delta\gamma_{22} = -\frac{1}{2}\frac{\Re\,(m_{21}\Gamma^{\rm LOY}_{12})}{m_0 - \mu}, \qquad (69)$$

$$\Delta\gamma_{12} = (\Delta\gamma_{21})^* = -\frac{1}{4}\frac{m_{12}}{m_0 - \mu}(\Gamma^{\rm LOY}_{11} + \Gamma^{\rm LOY}_{22}) \equiv -\frac{1}{2}\frac{m_{12}}{m_0 - \mu}\,\Gamma^{\rm LOY}_0; \qquad (70)$$

$\Re\,(z)$ denotes the real part of z, $m_{jk} \equiv H_{jk}$ $(j,k = 1,2)$, $m_0 \equiv H_{11} = H_{22}$, $\Gamma^{\rm LOY}_0 \equiv \Gamma^{\rm LOY}_{11} = \Gamma^{\rm LOY}_{22}$ and μ denotes the mass of the decay products. The last formula was obtained assuming that $|m_{12}| \equiv |H_{12}| \ll (m_0 - \mu)$.

Let us to assume for a moment that the order of the factor $\frac{1}{2}\frac{|m_{12}|}{m_0-\mu} \equiv \frac{1}{2}\frac{|\langle \mathbf{1}|H_{\rm I}|\mathbf{2}\rangle|}{m_0-\mu}$ appearing in (69) and (70) is the same as the ratio of a typical matrix element of $H_{\rm I} \equiv H_{\rm Weak}$ versus that of $H_{\rm Strong} \equiv H^{(0)}$; then (see [19], p. 352, (15.47))

$$\frac{1}{2}\frac{|\langle \mathbf{1}|H_{\rm I}|\mathbf{2}\rangle|}{m_0 - \mu} \sim \frac{1}{2}\frac{H_{\rm Weak}}{H_{\rm Strong}} \sim \frac{1}{2}\frac{G_{\rm F} m_p^2}{4\pi} \sim 10^{-7}, \qquad (71)$$

where $G_{\rm F}$ is the Fermi constant; we have natural units, $\hbar = c = 1$, and m_p is the mass of the proton. Thus one finds

$$|\Delta\gamma_{12}| \sim 10^{-7}\,\Gamma^{\rm LOY}_0, \qquad (72)$$

and

$$|\Delta\gamma_{11}| \le \frac{1}{2}\frac{|m_{12}|}{m_0 - \mu}\,|\Gamma^{\rm LOY}_{12}| \sim 10^{-7}|\Gamma^{\rm LOY}_{12}|. \qquad (73)$$

Note that the above estimations are rather unrealistic because, in fact, the weak interactions, $H_{\rm Weak}$, cannot induce first order $|\Delta S| = 2$ transitions in neutral kaon and similar complexes. So, we should have $\langle \mathbf{1}|H_{\rm I}|\mathbf{2}\rangle = 0$ in the case of $H_{\rm I} = H_{\rm Weak}$. More realistic estimations one obtains taking $H_{\rm I} \equiv H_{\rm SW}$, where $H_{\rm SW}$ denotes the hypothetical superweak interactions [19,37,38]. For such interactions the first order $|\Delta S| = 2$ transitions are allowed in the $K_0, \overline{K}_0$ subsystem, that is, possibly $\langle \mathbf{1}|H_{\rm I}|\mathbf{2}\rangle = \langle \mathbf{1}|H_{\rm SW}|\mathbf{2}\rangle \neq 0$. So one needs the estimation of the ratio $\frac{H_{\rm SW}}{H_{\rm Strong}}$ instead of $\frac{H_{\rm Weak}}{H_{\rm Strong}}$. Such an estimation can be found replacing $G_{\rm F}$ in (71) by $G_{\rm SW} = gG_{\rm F}$, where, according to [37,38], $g \sim 10^{-10} \div 10^{-11}$. This yields

$$|\Delta\gamma_{12}| \sim 10^{-17}\,\Gamma^{\rm LOY}_0, \quad |\Delta\gamma_{11}| \le 10^{-17}|\Gamma^{\rm LOY}_{12}| \qquad (74)$$

instead of (72) and (73). These estimations show that possible deviations from the LOY predictions are much too small to be observed with the present experiments. Nevertheless such deviations exist and lead to the non-zero effects of type (50), (69) and (70).

One of the Sylvester theorems states that a symmetric matrix defines a positive defined bilinear hermitian form if and only if all its angular minor determinants are positive (see, e.g., [39]). So, from the Sylvester theorem it follows that the matrix $\Delta\gamma$ can be positive definite if and only if $\Delta\gamma_{11} > 0$ and $\det\Delta\gamma > 0$ hold. We have

$$\det\Delta\gamma = \frac{1}{16}\frac{1}{(m_0 - \mu)^2} \times \left[4\,\left(\Re\,\left(m_{21}\,\Gamma^{\rm LOY}_{12}\right)\right)^2 - |m_{12}|^2\left(\Gamma^{\rm LOY}_{11} + \Gamma^{\rm LOY}_{22}\right)^2\right] \equiv \frac{1}{4}\frac{|m_{12}|^2}{(m_0 - \mu)^2}\left[\left(\Re\left(\frac{m_{21}}{|m_{12}|}\Gamma^{\rm LOY}_{12}\right)\right)^2 - (\Gamma^{\rm LOY}_0)^2\right], \qquad (75)$$

in the case considered. Now if we assume that $\Gamma^{\rm LOY}$ is positive definite, which is equivalent to the assumptions that $|\Gamma^{\rm LOY}_{12}|^2 \le \Gamma^{\rm LOY}_{11}\,\Gamma^{\rm LOY}_{22}$ and $\Gamma^{\rm LOY}_{11} > 0$, then $\det\Delta\gamma \le 0$. So, if in this case $\Gamma^{(0)} = \Gamma^{\rm LOY}$ is positive definite, then the matrix $\Delta\gamma$ cannot be positive definite. Therefore the matrix $\Gamma^{\rm FL} = \Gamma^{(0)} + \Delta\gamma$ need not be positive definite even in the case of $\Delta\gamma_{jk}$ $(j,k = 1,2)$ given by the estimation (74). This means that in such a model the hermitian form (63) need not fulfill the requirements of the scalar product and thus inequalities of type (64)–(67) cannot be considered as definitely valid.

Similar considerations lead to the conclusion that in the general case (50), the matrix $\Delta\Gamma$ need not be positive. Thus inequalities of type (64)–(67) may not be valid in the case of the relation (48).

From the above considerations the following conclusions follow.

If searching for the properties of neutral kaon and similar complexes one is going to use the estimations of type (67), one always should verify whether the matrix Γ is positive definite or not. The inequality (67) is true only for positive definite Γ. Of course, one always expects and assumes that Γ should be positive defined. Nevertheless we should remember that our assumptions or expectations can never replace an inspection or a rigorous proof.

From the relations (68)–(75) it follows that if there exist interactions in the system considered leading to matrix elements $\langle \mathbf{1}|H|\mathbf{2}\rangle \neq 0$, that is, if there exist interactions allowing for the first order $|\Delta S| = 2$ transitions, then the matrix Γ calculated within the more accurate approximation than $H_{\rm LOY}$ need not be positive definite. This means that in such a case conclusions following from the inequality of type (67) cannot be considered as ultimate. The same concerns tests for the existence of such interactions based on this inequality.

The standard derivation of the BS relation makes essential use of the assumption of the exponential form of the decay law of the neutral kaons. As it was mentioned earlier such an assumption [25] and other inconsistencies of this derivation are the cause that all tests of CPT-invariance based on the BS relation (1) cannot be considered as crucial.

Note that the BS relation (19) and the inequality (67) only contains quantities appearing in the eigenvalue equations (2) for the effective Hamiltonian $H_{||}$. Therefore one can assume that the real properties of the subsystems considered (like the neutral kaon complex, et cetera), will be described by solutions of the eigenvalue problem for the exact $H_{||}$. In other words, the result (67) means that one can expect the following. Estimations of parameters describing the neutral kaon complex performed within the use of the BS relation (19) and the inequality (67) describe real properties of this complex only if the quantities, which one inserts there, are extracted directly from the experiments and the positivity of Γ is rigorously proved. Of course these experiments must be designed in such a manner that the interpretation of the results of these tests is independent of the approximation used to describe the system under investigations. This means that, e.g., the parameters of the type γ_s, γ_l or $\Gamma_{l(s)}$, Γ_{jk}, cannot be extracted using the relations of the type (62). If one is unable to realize the test in such a manner then the interpretation of its results based on the BS relation (19) need not reflect the real properties of the system under investigation. There is the following reason for such a conclusion. Simply comparing the form of the formulae (25), (26) and (62) with expressions (45), (48) and (50) obtained within the more accurate approximation one finds that the real structure of the processes and interactions in the subsystem under investigation can be more complicated than it follows from the standard formulae (25), (26) and (62). All this has an effect on the real, measurable values of parameters describing the considered system. The BS relation in its original form (1) and also the LOY treatment of this problem are unable to correctly reflect all complicated processes of this kind.

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